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MONOTONE ITERATION SCHEME FOR IMPULSIVE THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS WITH QUADRATIC CONVERGENCE

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MONOTONE ITERATION SCHEME FOR IMPULSIVE THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS WITH QUADRATIC CONVERGENCE

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Abstract. In this paper, we consider an impulsive nonlinear second order ordinary differential equation with nonlinear three-point boundary conditions and develop a monotone iteration scheme by relaxing the convexity assumption on the function involved in the differential equation and the concavity assumption on nonlinearities in the boundary conditions. In fact, we obtain monotone sequences of iterates (approximate solutions) converging quadratically to the unique solution of the impulsive three-point boundary value problem.

1. Introduction

In recent years, a number of research papers have dealt with dynamical systems with impulse effect as a class of general hybrid systems. Impulsive hybrid systems are composed of some continuous variable dynamic systems along with certain reset maps that define impulsive switching among them. It is the switching that resets the modes and changes the continuous state of the system. There are three classes of impulsive hybrid systems, namely, impulsive differential systems [33, 41], sampled data or digital control system [30, 44] and impulsive switched system [19, 24]. Applications of such systems include air traffic management [43], automotive control [5, 9], real-time software verification [6], transportation systems [37, 45], manufacturing [39], mobile robotics [10], process industry [25], etc. In fact, hybrid systems have a central role in embedded control systems that interact with the physical world. Using hybrid models, one may represent time and event-based behaviors more accurately so as to meet challenging design requirements in the design of control systems for problems such as cut-off control and idle speed control of the engine. For more details, see [7] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for

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the better understanding of several real world problems in biology, physics, engineering, etc. Thus, the theory of impulsive differential equations is much richer than the corresponding theory of ordinary differential equations without impulse effects since a simple impulsive differential equation may exhibit several new phenomena such as rhythmical beating, merging of solutions and noncontinuability of solutions. For the theoretical aspect of impulsive differential equations, we refer the reader to the references [3, 4, 8, 13, 14, 16, 21, 33, 36, 40, 41, 47] whereas the applications of impulsive differential equations can be found in [15, 18, 26, 42].

Multi-point nonlinear boundary value problems which take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [22, 27-29]. In this paper, we develop an extended method of quasilinearization (the generalized quasilinearization) for a nonlinear impulsive three-point boundary value problem given by

$$x''(t) = f(t, x(t), x'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m, \tag{1.1}$$

$$px(0) - qx'(0) = g_1(x(\frac{1}{2})), \quad px(1) + qx'(1) = g_2(x(\frac{1}{2})), \tag{1.2}$$

and for $k = 1, 2, \ldots, m$,

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = v_k(x(t_k), x'(t_k)), \tag{1.3}$$

where $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous, $p,\ q$ are chosen appropriately with $p > 1$ and $g_{1,2} : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, $u_k \in \mathbb{R}$ and $v_k : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous for $k = 1, 2, \ldots, m$. This work is motivated by Doddaballapur, Eloe and Zhang [17] who discussed the method of quasilinearization for two-point boundary value problems with impulse. The origin of the quasilinearization lies in the theory of dynamic programming [11, 12]. This method applies to semilinear equations with convex or concave nonlinearities and generates a monotone iteration scheme whose iterates converge quadratically to a unique solution of the given problem. The nineties brought new dimensions to this technique when Lakshmikantham [31, 32] generalized the method of quasilinearization by relaxing the concavity/convexity assumption. This development proved to be of immense value and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, see [1, 22, 23, 34, 35] and references therein. Some real-world applications of the quasilinearization technique can be found in [2, 38, 46]. In Section 2, we present some basic results which play an important role to prove the main result of the paper. In Section 3, we apply the method of generalized quasilinearization to obtain monotone sequences of upper and lower solutions that converge quadratically to the unique solution of the problem (1.1)-(1.3).

2. Preliminary results

Let $PC[0,1]$ denote the class of piecewise continuous functions on $[0,1]$ and $PC^1[0,1]$ denote the class of functions x such that $x \in PC[0,1]$ and $x' \in PC[0,1]$. Define an appropriate Banach space B by

$$B = \{x \in PC^1[0,1] : x^{(i)}|_{[t_k,t_{k+1}]} \in C^i[t_k,t_{k+1}],\ k = 0,1,\dots,m,\ i = 0,1\},$$

with

$$\|x\|_B = \max_{k=0,1,\dots,m} \|x\|_k \text{ and } \|x\|_k = \max_{i=0,1} \sup_{t_k \le t \le t_{k+1}} |x^i(t)|.$$

For the fixed impulsive points t_k satisfying $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, we define the impulse by $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with the convention $x(t_k) = x(t_k^-)$, $k = 1,\dots,m$.

We say that $\alpha \in B$ is a lower solution of the BVP with impulse (1.1)-(1.3) if

$$\alpha''(t) \ge f(t,\alpha(t),\alpha'(t)),\ t_k < t < t_{k+1},\ k = 0,1,\dots,m,$$

$$p\alpha(0) - q\alpha'(0) \le g_1(\alpha(\frac{1}{2})), \quad p\alpha(1) + q\alpha'(1) \le g_2(\alpha(\frac{1}{2})),$$

and for $k = 1,2,\dots,m$,

$$\Delta\alpha(t_k) = u_k,$$

$$\Delta\alpha'(t_k) \ge v_k(\alpha(t_k),\alpha'(t_k)).$$

Analogously, $\beta \in B$ is an upper solution of the BVP with impulse (1.1)-(1.3) if

$$\beta''(t) \le f(t,(\beta(t),\beta'(t)),\ t_k < t < t_{k+1},\ k = 0,1,\dots,m,$$

$$p\beta(0) - q\beta'(0) \ge g_1(\beta(\frac{1}{2})), \quad p\beta(1) + q\beta'(1) \ge g_2(\beta(\frac{1}{2})),$$

and for $k = 1,2,\dots,m$,

$$\Delta\beta(t_k) = u_k,$$

$$\Delta\beta'(t_k) \le v_k(\beta(t_k),\beta'(t_k)).$$

Finally, we define a partial order on B as

$$\alpha|_{[t_k,t_{k+1}]}(t) \le \beta|_{[t_k,t_{k+1}]}(t),\ t_k < t < t_{k+1},\ k = 0,1,\dots,m.$$

As argued in [17], x is a solution of BVP with impulse (1.1)-(1.3) if and only if $x \in B$ and $Tx = x$, where T is a fixed point operator defined by

$$Tx(t) = p(t) + I(t,x) + \int_0^1 G(t,s)f(s,x(s),x'(s))ds, \tag{2.1}$$

where

$$p(t) = g_1(x(\frac{1}{2}))(\frac{-t}{p+2q} + \frac{p+q}{p^2+2pq}) + g_2(x(\frac{1}{2}))(\frac{t}{p+2q} + \frac{q}{p^2+2pq}),$$

$$G(t,s) = \frac{1}{(p^2+2pq)} \begin{cases} (pt+q)(p(s-1)-q), & \text{if } 0 \le t \le s \le 1, \\ (p(t-1)-q)(ps+q), & \text{if } 0 \le s \le t \le 1, \end{cases}$$

and $I(t,x) = \sum_{k=1}^{m} I_k(t,x)$ with

$$I_k(t,x) = \frac{1}{p^2+2pq} \begin{cases} (pt+q)[-u_k p + (p(t_k-1)-q)v_k(x(t_k),x'(t_k))], & 0 \le t \le t_k, \\ (p(t-1)-q)[-u_k p + (pt_k+q)v_k(x(t_k),x'(t_k))], & t_k \le t \le 1. \end{cases}$$

Now, we are in a position to present some basic results. For the sake of completeness, we provide the proof of these theorems although the method of proof is similar to the one employed in [17].

Theorem 2.1. *Assume that*

(i) $\alpha,\ \beta \in B$ *are lower and upper solutions of the BVP with impulse* (1.1)-(1.3) *respectively.*

(ii) $f, f_x \in C([0,1]\times\mathbb{R}^2)$ *such that* $f_x(t,x,y) > 0$ *and* $v_k \in C^1(\mathbb{R}^2)$ *with* $v_{kx}(x,y) > 0,\ v_{ky}(x,y) > 0,\ (x,y)\in\mathbb{R}^2$ *for* $k = 1,2,\ldots,m$.

(iii) g_i *are continuous on* $\mathbb{R}$ *and satisfy one-sided Lipschitz condition*:

$$g_i(x) - g_i(y) \le L_i(x-y),\ 0 \le L_i < 1,\ i = 1,2.$$

Then $\alpha(t) \le \beta(t)$.

Proof. Let us set $w(t) = \alpha(t) - \beta(t)$. For the sake of contradiction, we assume that w is positive on $[0,1]$, that is, $\alpha(t) > \beta(t)$. By definition of upper and lower solutions and using (iii), we find that

$$\begin{aligned} pw(0) - qw'(0) &\le g_1(\alpha(\tfrac{1}{2})) - g_1(\beta(\tfrac{1}{2})) \\ &\le L_1(\alpha(\tfrac{1}{2}) - \beta(\tfrac{1}{2})) \\ &< \alpha(\tfrac{1}{2}) - \beta(\tfrac{1}{2}) = w(\tfrac{1}{2}), \end{aligned}$$

From this inequality, it follows that $w(t)$ does not have a local positive maximum at $\tau = 0$ as $w'(0) = 0$ implies that $w(0) < w(\frac{1}{2})$ for $p > 1$. Similarly, it can be shown that $w(t)$ cannot have a local positive maximum at $\tau = 1$. Thus, w has a positive maximum at some $\tau \in (0,1)$. Let us assume that $\tau \in \bigcup_{k=0}^{m}(t_k,t_{k+1})$. Then $w''(\tau) \le 0$ and $\alpha'(\tau) = \beta'(\tau)$. Since α and β are lower and upper solutions of the BVP with impulse (1.1)-(1.3) and $f_x > 0$, it follows that

$$w''(\tau) = \alpha''(\tau) - \beta''(\tau) \ge f(\tau,\alpha(\tau),\alpha'(\tau)) - f(\tau,\beta(\tau),\beta'(\tau)) > 0.$$

This provides a contradiction and so, $\tau \notin \bigcup_{k=0}^{m}(t_k,t_{k+1})$.

Now, we assume that $\tau = t_k$ for some $k \in \{1,2,\ldots,m\}$. Then, by Taylor's theorem, $w'(t_k^-) \ge 0$ and $w'(t_k^+) \le 0$, or $\Delta w'(t_k) \le 0$ and

$$\alpha'(t_k^-) = \alpha'(t_k) \ge \beta'(t_k) = \beta'(t_k^-).$$

In view of the fact that $v_{kx}(x,y) > 0,\ v_{ky}(x,y) > 0,\ (x,y) \in \mathbb{R}^2$, it follows that

$$\Delta w'(t_k) = \Delta\alpha'(t_k) - \Delta\beta'(t_k) \geq v_k(\alpha(t_k), \alpha'(t_k)) - v_k(\beta(t_k), \beta'(t_k)) > 0,$$

which is a contradiction and thus $\tau \notin \{t_1, \ldots, t_m\}$. Hence, we conclude that $\alpha(t) \leq \beta(t),\ t \in [0,1]$. This completes the proof. □

Theorem 2.2. *Assume that* $f \in C([0,1] \times \mathbb{R}^2),\ z_k \in C(\mathbb{R}^2),\ k = 1, \ldots, m$ *and each* $z_k(x,y)$ *is monotone increasing in* y *for fixed* x*. Further, each solution of* $x''(t) = f(t, x(t), x'(t))$ *extends to* $[0,1]$ *or becomes unbounded on its maximal interval of convergence. Let* $\alpha,\ \beta$ *be lower and upper solutions respectively of the BVP*

$$x''(t) = f(t, x(t), x'(t)),\ t_k < t < t_{k+1}, \tag{2.2}$$

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = z_k(x(t_k), x'(t_k)),\ k = 1, 2, \ldots, m, \tag{2.3}$$

with boundary conditions (1.2) *such that* $\alpha \leq \beta$*. Further,* g_i $(i = 1,2)$ *in* (1.2) *are continuous on* $\mathbb{R}$ *and satisfy one-sided Lipschitz condition. Then, there exists a solution* x *of the BVP with impulse* (2.2)-(2.3), (1.2) *satisfying* $\alpha \leq x \leq \beta$.

Proof. We define

$$\hat{f}(t,x,y) = \begin{cases} f(t, \beta(t), y) + \frac{x - \beta(t)}{1 + (x - \beta(t))}, & \text{if } x(t) > \beta(t), \\ f(t,x,y), & \text{if } \alpha(t) \leq x(t) \leq \beta(t), \\ f(t, \alpha(t), y) + \frac{x - \alpha(t)}{1 + |x - \alpha(t)|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$\hat{v}_k(x,y) = \begin{cases} z_k(\beta(t_k), y) + \frac{x - \beta(t_k)}{1 + (x - \beta(t_k))}, & \text{if } x > \beta(t_k), \\ z_k(x,y), & \text{if } \alpha(t_k) \leq x \leq \beta(t_k), \\ z_k(\alpha(t_k), y) + \frac{x - \alpha(t_k)}{1 + |x - \alpha(t_k)|}, & \text{if } x < \alpha(t_k) \end{cases}$$

for $k = 1, 2, \ldots, m$ and

$$G_i(x) = \begin{cases} g_i(\beta(\frac{1}{2})), & \text{if } x > \beta(\frac{1}{2}), \\ g_i(x), & \text{if } \alpha(\frac{1}{2}) \leq x \leq \beta(\frac{1}{2}), \\ g_i(\alpha(\frac{1}{2})), & \text{if } x < \alpha(\frac{1}{2}) \end{cases}$$

for $i = 1, 2$. Let $N > 0$ be such that $|\alpha'(t)| \leq N,\ |\beta'(t)| \leq N,\ t \in [t_k, t_{k+1}],\ k = 0, 1, \ldots, m$.

For each positive integer l, define

$$f_l(t,x,y) = \begin{cases} \hat{f}(t, x, N + l), & \text{if } y > N + l, \\ \hat{f}(t,x,y), & \text{if } |y| \leq N + l, \\ \hat{f}(t, x, -(N + l)), & \text{if } y < -(N + l), \end{cases}$$

and

$$v_{kl}(x,y) = \begin{cases} \hat{v}_k(x, N + l), & \text{if } y > N + l, \\ \hat{v}_k(x,y), & \text{if } |y| \leq N + l, \\ \hat{v}_k(x, -(N + l)), & \text{if } y < -(N + l). \end{cases}$$

Notice that f_l, G_i and v_{kl} are bounded and continuous on their respective domains. With a standard application of the Schauder fixed point theorem to the operator T, defined by (2.1), we can obtain a solution, $x_l \in B$, to the BVP with impulse (2.2)-(2.3), (1.2) with $f = f_l$, each $G_i = g_i$ and each $v_k = v_{kl}$.

Now, we want to show that each solution x_l satisfies $\alpha \le x_l \le \beta$. We set $r(t) = x_l(t) - \beta(t)$ and assume that $r(t)$ has a positive maximum at $\tau \in (0,1)$ as in the proof of Theorem 2.1. If $\tau \in \bigcup_{k=0}^{m} (t_k, t_{k+1})$, then $r''(\tau) \le 0$, that is, $x_l''(\tau) \le \beta''(\tau)$, and $|x_l'(\tau)| = |\beta'(\tau)| \le N < N + l$. Thus

$$\begin{aligned} r''(\tau) &= x_l''(\tau) - \beta''(\tau) \\ &\ge f(\tau, \beta(\tau), x_l'(\tau)) + \frac{x_l(\tau) - \beta(\tau)}{1 + (x_l(\tau) - \beta(\tau))} - f(\tau, \beta(\tau), \beta'(\tau)) \\ &= \frac{x_l(\tau) - \beta(\tau)}{1 + (x_l(\tau) - \beta(\tau))} > 0, \end{aligned}$$

which is a contradiction. Now, suppose that $\Delta r'(\tau) \le 0$ for $\tau = t_k$, $k = 1, 2, \ldots, m$. Since each $z_k(x, y)$ is monotone increasing in y for fixed x, it follows that each $v_{kl}(x, y)$ is monotone increasing in y for fixed x. Moreover, we note that each $v_{kl}(\beta(t_k), \beta'(t_k)) = z_k(\beta(t_k), \beta'(t_k))$. Thus,

$$\begin{aligned} \Delta r'(t_k) &\ge v_{kl}(\beta(t_k), x_l'(t_k)) - v_{kl}(\beta(t_k), \beta'(t_k)) + \frac{x_l(t_k) - \beta(t_k)}{1 + (x_l(t_k) - \beta(t_k))} \\ &\ge \frac{x_l(t_k) - \beta(t_k)}{1 + (x_l(t_k) - \beta(t_k))} > 0, \end{aligned}$$

which is also a contradiction. Therefore, $r(t) \le 0$ or $x_l \le \beta$. On the same pattern, we can prove that $\alpha(t) \le x_l(t)$.

For each l, there exists $t_l \in [0, t_1]$ such that

$$t_1 |x_{kl}'(t_l)| = |x_{kl}(t_1) - x_{kl}(0)| \le \max\{|\beta(0) - \alpha(t_1)|,\ |\beta(t_1) - \alpha(0)|\}.$$

Thus, the sequences $\{x_{kl}(t_l)\}$ and $\{x_{kl}'(t_l)\}$ are bounded. Now, applying the Kamke convergence theorem for solutions of initial value problems, we obtain a subsequence of $\{x_{kl}\}$ which converges to a solution of $x''(t) = \hat{f}(t, x(t), x'(t))$ on a maximal subinterval of $[0, t_1]$. Clearly, $\alpha(t) \le x(t) \le \beta(t)$ and the solutions of $x''(t) = f(t, x(t), x'(t))$ extend to all of $[0,1]$ or become unbounded implying that $x''(t) = \hat{f}(t, x(t), x'(t))$ on $[0, t_1]$.

Now, we apply the impulse $\Delta x'(t_k) = z_k(x(t_k), x'(t_k))$ at t_1. Applying the Kamke theorem on the subsequence obtained earlier, we can employ $t_1 = t_l$ for each l. Thus, we can obtain a further subsequence which converges to a solution x of $x''(t) = \hat{f}(t, x(t), x'(t))$ on $(0, t_1) \bigcup (t_1, t_2)$ such that x satisfies the impulse $\Delta x'(t_k) = z_k(x(t_k), x'(t_k))$ at t_1. This procedure can be continued inductively by first applying the impulse at each t_j and then applying the Kamke convergence theorem on the subinterval (t_j, t_{j+1}). Hence $\alpha(t) \le x(t) \le \beta(t)$ and $\hat{f}(t, x(t), x'(t)) = f(t, x(t), x'(t))$. This completes the proof. $\square$

Remark. The simplified version of the condition that each solution of $x''(t) = f(t, x(t), x'(t))$ extends to $[0,1]$ or becomes unbounded on its maximal interval of convergence is that f satisfies a Nagumo condition, that is, for each $M > 0$, there exists a positive continuous function h_M on $[0, \infty)$ such that $|f(t, x, x')| \leq h_M(|x'|)$ for all $(t, x, x') \in [0,1] \times [-M, M] \times \mathbb{R}$ and

$$\int_0^\infty (s/h_M(s))ds = \infty.$$

3. Iteration scheme with quadratic convergence

Theorem 3.1. *Assume that*

($\mathbf{A_0}$) $\alpha_0, \beta_0 \in B$ *are lower and upper solutions of the BVP with impulse* (1.1)-(1.3) *respectively..*

($\mathbf{A_1}$) $f(t,x,y) \in C([0,1] \times \mathbb{R}^2)$ *be such that* $f_x(t,x,y) > 0$, $\frac{\partial^2}{\partial x^2}(f(t,x,y) + \phi_1(t,x,y)) \geq 0$, *where* $\frac{\partial^2}{\partial x^2}\phi_1(t,x,y) \geq 0$ *for some continuous function* $\phi_1(t,x,y)$. *Moreover, f satisfies a Nagumo condition in x'.*

($\mathbf{A_2}$) *For* $i = 1, 2$, $g_i(x), g_i'(x)$ *are continuous on* $\mathbb{R}$ *with* $0 \leq g_i' \leq 1$ *and* $g_i''(x) + \psi_i''(x) \leq 0$ *with* $\psi_i'' \leq 0$ *for some continuous functions* $\psi_i(x)$ *on* $\mathbb{R}$.

($\mathbf{A_3}$) $v_k \in C^1(\mathbb{R}^2)$ *such that* $v_{kx}(x,y) > 0$, $v_{ky}(x,y) > 0$, $(x,y) \in \mathbb{R}^2$ *and* $v_k'' \in C(\mathbb{R}^2)$ *for* $k = 1, 2, \ldots, m$.

Then there exists monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$, *which converge in B to a unique solution x of the BVP with impulse* (1.1)-(1.3) *and the convergence is quadratic.*

Proof. Let $F(t,x) : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that F, F_x and F_{xx} are continuous on $[0,1] \times \mathbb{R}$ and it follows from (A_1) that $F_{xx} \geq 0$ on $[0,1] \times \mathbb{R}$, where we have set [23]

$$f(t, x_1, x_2) = F(t, x_1) - \phi_1(t, x_1, x_2).$$

Applying the mean value theorem on $F(t, x_1)$ gives

$$F(t, x_1) \geq F(t, y_1) + F_x(t, y_1)(x_1 - y_1),$$

which together with the definition of $F(t, x_1)$ takes the form

$$f(t, x_1, x_2) \geq f(t, y_1, y_2) + F_x(t, y_1)(x_1 - y_1) - \phi_1(t, x_1, x_2) + \phi_1(t, y_1, y_2) \tag{3.1}$$

for $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

We define $G_i : \mathbb{R} \longrightarrow \mathbb{R}$ by $G_i(x) = g_i(x) + \psi_i(x)$, $i = 1, 2$. From (A_2), it follows that

$$g_i(x) \leq g_i(y) + G_i'(y)(x - y) + \psi_i(y) - \psi_i(x), i = 1, 2, \; x, y \in \mathbb{R}. \tag{3.2}$$

For each $k = 1, \ldots, m$, let $V_k(x) : \mathbb{R} \longrightarrow \mathbb{R}$ be such that V_k, V_k', V_k'' are continuous on $\mathbb{R}$ with $V_k''(x) \geq 0$, $x \in \mathbb{R}$. Let us set $\phi_k(x_1, x_2) = V_k(x_1) - v_k(x_1, x_2)$

on $\mathbb{R}^2$. By the mean value theorem, we have $V_k(x_1) \geq V_k(y_1) + V_k'(y_1)(x_1 - y_1)$ for $x_1, y_1 \in \mathbb{R}$, which can alternatively be written as

$$v_k(x_1, x_2) \geq v_k(y_1, y_2) + V_k'(y_1)(x_1 - y_1) - \phi_k(x_1, x_2) + \phi_k(y_1, y_2) \tag{3.3}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Now, we define

$$\begin{aligned}
Q(t, x_1, x_2; \alpha_0, \beta_0, \alpha_0') &= f(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(x_1 - \alpha_0(t)) \\
&\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \alpha_0(t), \alpha_0'(t)), \\
J(t, x_1, x_2; \beta_0, \beta_0') &= f(t, \beta_0(t), \beta_0'(t)) + F_x(t, \beta_0(t))(x_1 - \beta_0(t)) \\
&\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \beta_0(t), \beta_0'(t)), \\
h_k(x_1, x_2; \alpha_0, \beta_0, \alpha_0') &= v_k(\alpha_0(t_k), \alpha_0'(t_k)) + V_k'(\beta_0(t_k))(x_1 - \alpha_0(t_k)) \\
&\quad - \phi_k(x_1, x_2) + \phi_k(\alpha_0(t_k), \alpha_0'(t_k)), \\
H_k(x_1, x_2; \beta_0, \beta_0') &= v_k(\beta_0(t_k), \beta_0'(t_k)) + V_k'(\beta_0(t_k))(x_1 - \beta_0(t_k)) \\
&\quad - \phi_k(x_1, x_2) + \phi_k(\beta_0(t_k), \beta_0'(t_k)), \\
r_i(x(\frac{1}{2}); \alpha_0, \beta_0) &= g_i(\alpha_0(\frac{1}{2})) + G_i'(\beta_0(\frac{1}{2}))(x(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \\
&\quad + \psi_i(\alpha_0(\frac{1}{2})) - \psi_i(x(\frac{1}{2})), i = 1, 2, \\
R_i(x(\frac{1}{2}); \beta_0) &= g_i(\beta_0(\frac{1}{2})) + G_i'(\beta_0(\frac{1}{2}))(x(\frac{1}{2}) - \beta_0(\frac{1}{2})) \\
&\quad + \psi_i(\beta_0(\frac{1}{2})) - \psi_i(x(\frac{1}{2})), i = 1, 2.
\end{aligned}$$

Observe that

$$Q(t, \alpha_0(t), \alpha_0'(t); \alpha_0, \beta_0, \alpha_0') = f(t, \alpha_0(t), \alpha_0'(t)),$$

$$f(t, x_1, x_2) \geq Q(t, x_1, x_2; \alpha_0, \beta_0, \alpha_0'). \tag{3.4}$$

$$r_i(\alpha_0(\frac{1}{2}); \alpha_0, \beta_0) = g_i(\alpha_0(\frac{1}{2})),\ \ g_i(x) \leq r_i(x(\frac{1}{2}); \alpha_0, \beta_0)\ \ i = 1, 2. \tag{3.5}$$

$$h_k(\alpha_0(t_k), \alpha_0'(t_k); \alpha_0, \beta_0, \alpha_0') = v_k(\alpha_0(t_k), \alpha_0'(t_k)),$$

$$v_k(x_1(t), x_2(t)) \geq h_k(x_1, x_2; \alpha_0, \beta_0, \alpha_0'). \tag{3.6}$$

Also

$$J(t, \beta_0(t), \beta_0'(t); \beta_0, \beta_0') = f(t, \beta_0(t), \beta_0'(t)),\ \ f(t, x_1, x_2) \leq J(t, x_1, x_2; \beta_0, \beta_0'). \tag{3.7}$$

$$R_i(\beta_0(\frac{1}{2}); \beta_0) = g_i(\beta_0(\frac{1}{2})),\ \ g_i(x) \geq R_i(x(\frac{1}{2}); \beta_0),\ \ i = 1, 2. \tag{3.8}$$

$$H_k(\beta_0(t_k), \beta_0'(t_k); \beta_0, \beta_0') = v_k(\beta_0(t_k), \beta_0'(t_k)),$$

$$v_k(x_1(t), x_2(t)) \leq H_k(x_1, x_2; \beta_0, \beta_0'). \tag{3.9}$$

Consider the BVP with impulse

$$x''(t) = Q(t, x(t), x'(t); \alpha_0, \beta_0, \alpha_0'),\ t_k < t < t_{k+1},\ k = 0, \dots, m, \tag{3.10}$$

$$px(0) - qx'(0) = r_1(x(\frac{1}{2}); \alpha_0, \beta_0), \quad px(1) + qx'(1) = r_2(x(\frac{1}{2}); \alpha_0, \beta_0), \tag{3.11}$$

and for $k = 1, \dots, m$,

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = h_k(x(t_k), x'(t_k); \alpha_0, \beta_0, \alpha_0'). \tag{3.12}$$

We show that α_0 and β_0 are respectively lower and upper solutions of the BVP with impulse (3.10)-(3.12).

Using (3.4)-(3.6) together with the fact that α_0 is a lower solution of (1.1)-(1.3), we obtain

$$\begin{aligned}
\alpha_0''(t) &\geq f(t, \alpha_0(t), \alpha_0'(t)) \\
&= Q(t, \alpha_0(t), \alpha_0'(t); \alpha_0, \beta_0, \alpha_0'),\ t_k < t < t_{k+1}, k = 0, 1, \dots, m, \\
p\alpha_0(0) - q\alpha_0'(0) &\leq g_1(\alpha_0(\frac{1}{2})) = r_1(\alpha_0(\frac{1}{2}); \alpha_0, \beta_0), \\
p\alpha_0(1) + q\alpha_0'(1) &\leq g_2(\alpha_0(\frac{1}{2})) = r_2(\alpha_0(\frac{1}{2}); \alpha_0, \beta_0),
\end{aligned}$$

and for $k = 1, \dots, m$,

$$\Delta \alpha_0(t_k) = u_k,$$

$$\Delta \alpha_0'(t_k) \geq v_k(\alpha_0(t_k),\ \alpha_0'(t_k)) = h_k(\alpha_0(t_k), \alpha_0'(t_k); \alpha_0, \beta_0, \alpha_0'),$$

which implies that α_0 is a lower solution of (3.10)-(3.12).

Using (3.1) and the fact that β_0 is an upper solution of (1.1)-(1.3), we get

$$\begin{aligned}
\beta_0''(t) &\leq f(t, \beta_0(t), \beta_0'(t)) \\
&\leq f(t, \alpha_0(t), \alpha_0'(t)) - F_x(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\
&\quad +\phi_1(t, \alpha_0(t), \alpha_0'(t)) - \phi_1(t, \beta_0(t), \beta_0'(t)) \\
&= Q(t, \beta_0(t), \beta_0'(t); \alpha_0, \beta_0, \alpha_0'),\ t_k < t < t_{k+1},\ k = 0, 1, \dots, m.
\end{aligned}$$

Using the mean value theorem and the non-increasing property of G_1', we have

$$\begin{aligned}
&g_1(\beta_0(\frac{1}{2})) - r_1(\beta_0(\frac{1}{2}); \alpha_0, \beta_0) \\
= \ &g_1(\beta_0(\frac{1}{2})) - g_1(\alpha_0(\frac{1}{2})) - G_1'(\beta_0(\frac{1}{2}))(\beta_0(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \\
&-\psi_1(\alpha_0(\frac{1}{2})) + \psi_1(\beta_0(\frac{1}{2})) \\
= \ &G_1(\beta_0(\frac{1}{2})) - G_1(\alpha_0(\frac{1}{2})) - G_1'(\beta_0(\frac{1}{2}))(\beta_0(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \\
= \ &(G_1'(c_0) - G_1'(\beta_0(\frac{1}{2})))(\beta_0(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \geq 0,
\end{aligned}$$

$(\alpha_0(\frac{1}{2}) \leq c_0 \leq \beta_0(\frac{1}{2}))$ which yields

$$p\beta_0(0) - q\beta_0'(0) \geq g_1(\beta_0(\frac{1}{2})) \geq r_1(\beta_0(\frac{1}{2}); \alpha_0, \beta_0).$$

Similarly, it can be shown that

$$p\beta_0(1) + q\beta_0'(1) \geq r_2(\beta_0(\frac{1}{2}); \alpha_0, \beta_0).$$

In view of (3.3), we get

$$\begin{aligned}\Delta\beta_0'(t_k) &\leq v_k(\beta_0(t_k), \beta_0'(t_k)) \\ &\leq v_k(\alpha_0(t_k), \alpha_0'(t_k)) - V_k'(\beta_0(t_k))(\alpha_0(t_k) - \beta_0(t_k)) \\ &\quad + \phi_k(\alpha_0(t_k), \alpha_0'(t_k)) - \phi_k(\beta_0(t_k), \beta_0'(t_k)) \\ &= h_k(\beta_0(t_k), \beta_0'(t_k); \alpha_0, \beta_0, \alpha_0'),\ k = 1, \ldots, m.\end{aligned}$$

Thus, we conclude that β_0 is an upper solution of the BVP with impulse (3.10)-(3.12). Moreover, each h_k satisfies the hypotheses of Theorem 2.2 and a limit comparison implies that Q satisfies a Nagumo condition in x'. Hence, by Theorem 2.2, there exists a solution α_1 of the BVP with impulse (3.10)-(3.12) such that

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t). \tag{3.13}$$

Now, we consider the BVP with impulse

$$x''(t) = J(t, x(t), x'(t); \beta_0, \beta_0'),\ t_k < t < t_{k+1},\ k = 0, \ldots, m, \tag{3.14}$$

$$px(0) - qx'(0) = R_1(x(\frac{1}{2}); \beta_0), \quad px(1) + qx'(1) = R_2(x(\frac{1}{2}); \beta_0), \tag{3.15}$$

and for $k = 1, \ldots, m$,

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = H_k(x(t_k), x'(t_k); \alpha_0, \beta_0, \beta_0'). \tag{3.16}$$

Since J and each H_k satisfy the hypotheses of Theorem 2.2, we can repeat the earlier arguments to conclude that α_0 and β_0 are respectively lower and upper solutions of the BVP with impulse (3.14)-(3.16). Thus, by Theorem 2.2, there exists a solution β_1 of the BVP with impulse (3.14)-(3.16) such that

$$\alpha_0 \leq \beta_1 \leq \beta_0. \tag{3.17}$$

Next, we show that α_1 and β_1 are respectively lower and upper solutions of the BVP with impulse (1.1)-(1.3). Since α_1 is a solution of (3.10)-(3.12) such that

$\alpha_0 \le \alpha_1 \le \beta_0$, it follows from (A_1) and (3.1) that

$$\begin{aligned}
\alpha_1''(t) &= Q(t, \alpha_1(t), \alpha_1'(t); \alpha_0, \beta_0, \alpha_0') \\
&= f(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) \\
&\quad - \phi_1(t, \alpha_1(t), \alpha_1'(t)) + \phi_1(t, \alpha_0(t), \alpha_0'(t)) \\
&\ge f(t, \alpha_1(t), \alpha_1'(t)) + F_x(t, \alpha_1(t))(\alpha_0(t) - \alpha_1(t)) \\
&\quad + \phi_1(t, \alpha_1(t), \alpha_1'(t)) - \phi_1(t, \alpha_0(t), \alpha_0'(t)) \\
&\quad + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) - \phi_1(t, \alpha_1(t), \alpha_1'(t)) + \phi_1(t, \alpha_0(t), \alpha_0'(t)) \\
&= f(t, \alpha_1(t), \alpha_1'(t)) + (F_x(t, \beta_0(t)) - F_x(t, \alpha_1(t)))(\alpha_1(t) - \alpha_0(t)) \\
&\ge f(t, \alpha_1(t), \alpha_1'(t)), \ t \in \bigcup_{k=0}^{m} (t_k, t_{k+1}).
\end{aligned}$$

In view of non-increasing property of G_1', we get

$$\begin{aligned}
&\quad g_1(\alpha_1(\frac{1}{2})) - (p\alpha_1(0) - q\alpha_1'(0)) \\
&= g_1(\alpha_1(\frac{1}{2})) - r_1(\alpha_1(\frac{1}{2}); \alpha_0, \beta_0) \\
&= g_1(\alpha_1(\frac{1}{2})) - g_1(\alpha_0(\frac{1}{2})) - G_1'(\beta_0(\frac{1}{2}))(\alpha_1(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \\
&\quad - \psi_1(\alpha_0(\frac{1}{2})) + \psi_1(\alpha_1(\frac{1}{2})) \\
&= G_1(\alpha_1(\frac{1}{2})) - G_1(\alpha_0(\frac{1}{2})) - G_1'(\beta_0(\frac{1}{2}))(\alpha_1(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \\
&= (G_1'(c_1) - G_1'(\beta_0(\frac{1}{2})))(\alpha_1(\frac{1}{2}) - \alpha_0(\frac{1}{2})) \ge 0,
\end{aligned}$$

$(\alpha_0(\frac{1}{2}) \le c_1 \le \alpha_1(\frac{1}{2}))$ which yields $p\alpha_1(0) - q\alpha_1'(0) \le g_1(\alpha_1(\frac{1}{2}))$. Similarly, we have $p\alpha_1(1) + q\alpha_1'(1) \le g_2(\alpha_1(\frac{1}{2}))$. Using (3.3) and the nondecreasing property of V_k', we obtain

$$\begin{aligned}
&\quad \Delta\alpha_1'(t_k) \\
&= h_k(\alpha_1(t_k), \alpha_1'(t_k); \alpha_0, \beta_0, \alpha_0') \\
&= v_k(\alpha_0(t_k), \alpha_0'(t_k)) + V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) \\
&\quad -\phi_k(\alpha_1(t_k), \alpha_1'(t_k)) + \phi_k(\alpha_0(t_k), \alpha_0'(t_k)) \\
&\ge v_k(\alpha_1(t_k), \alpha_1'(t_k)) + V_k'(\alpha_1(t_k))(\alpha_0(t_k) - \alpha_1(t_k)) \\
&\quad +\phi_k(\alpha_1(t_k), \alpha_1'(t_k)) - \phi_k(\alpha_0(t_k), \alpha_0'(t_k)) \\
&\quad +V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) - \phi_k(\alpha_1(t_k), \alpha_1'(t_k)) + \phi_k(\alpha_0(t_k), \alpha_0'(t_k)) \\
&= v_k(\alpha_1(t_k), \alpha_1'(t_k)) + (V_k'(\beta_0(t_k)) - V_k'(\alpha_1(t_k)))(\alpha_1(t_k) - \alpha_0(t_k)) \\
&\ge v_k(\alpha_1(t_k), \alpha_1'(t_k)), \ k = 1, 2, \ldots, m.
\end{aligned}$$

Hence α_1 is a lower solution of (3.1.1)-(3.1.3). Similarly, for $t \in \bigcup_{k=0}^{m}(t_k, t_{k+1})$, it can be shown that

$$\beta_1''(t) \le f(t, \beta_1(t), \beta_1'(t)),$$
$$p\beta_1(0) - q\beta_1'(0) \ge g_1(\beta_1(\frac{1}{2})),\ p\beta_1(1) + q\beta_1'(1) \ge g_2(\beta_1(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\Delta\beta_1'(t_k) \le v_k(\beta_1(t_k), \beta_1'(t_k)).$$

Thus, β_1 is an upper solution of (3.1.1)-(3.1.3). Hence, by Theorem 2.1, we have

$$\alpha_1 \le \beta_1. \tag{3.18}$$

Combining (3.13), (3.17) and (3.18) yields

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t),\ t \in \bigcup_{k=0}^{m}(t_k, t_{k+1}).$$

Let us define the following sequences of functions $\{Q_l\}, \{J_l\}, \{r_{il}\}, \{R_{il}\}, \{h_{kl}\}$ and $\{H_{kl}\}$ by

$$\begin{aligned}
Q_l(t, x_1, x_2) &= Q(t, x_1, x_2; \alpha_l, \beta_l, \alpha_l') \\
&= f(t, \alpha_l(t), \alpha_l'(t)) + F_x(t, \beta_l(t))(x_1 - \alpha_l(t)) \\
&\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \alpha_l(t), \alpha_l'(t)), \\
J_l(t, x_1, x_2) &= J(t, x_1, x_2; \beta_l, \beta_l') = f(t, \beta_l(t), \beta_l'(t)) + F_x(t, \beta_l(t))(x_1 - \beta_l(t)) \\
&\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \beta_l(t), \beta_l'(t)), \\
h_{kl} &= h_k(x_1, x_2; \alpha_l, \beta_l, \alpha_l') \\
&= v_k(\alpha_l(t_k), \alpha_l'(t_k)) + V_k'(\beta_l(t_k))(x_1 - \alpha_l(t_k)) \\
&\quad - \phi_k(x_1, x_2) + \phi_k(\alpha_l(t_k), \alpha_l'(t_k)), \\
H_{kl} &= H_k(x_1, x_2; \beta_l, \beta_l') = v_k(\beta_l(t_k), \beta_l'(t_k)) + V_k'(\beta_l(t_k))(x_1 - \beta_l(t_k)) \\
&\quad - \phi_k(x_1, x_2) + \phi_k(\beta_l(t_k), \beta_l'(t_k)), \\
r_{il} &= r_i(x(\frac{1}{2}); \alpha_l, \beta_l) = g_i(\alpha_l(\frac{1}{2})) + G_i'(\beta_l(\frac{1}{2}))(x(\frac{1}{2}) - \alpha_l(\frac{1}{2})) \\
&\quad + \psi_i(\alpha_l(\frac{1}{2})) - \psi_i(x(\frac{1}{2})),\ i = 1, 2, \\
R_{il} &= R_i(x(\frac{1}{2}); \beta_l) = g_i(\beta_l(\frac{1}{2})) + G_i'(\beta_l(\frac{1}{2}))(x(\frac{1}{2}) - \beta_l(\frac{1}{2})) \\
&\quad + \psi_i(\beta_l(\frac{1}{2})) - \psi_i(x(\frac{1}{2})),\ i = 1, 2.
\end{aligned}$$

Now, by induction, it can be proved that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_l \le \alpha_{l+1} \le \beta_{l+1} \le \beta_l \le \cdots \le \beta_1 \le \beta_0.$$

For that, we consider the following BVPs with impulse

$$x''(t) = Q(t, x(t), x'(t); \alpha_l, \beta_l, \alpha_l'),\ t_k < t < t_{k+1},\ k = 0, \dots, m, \tag{3.19}$$

$$px(0) - qx'(0) = r_1(x(\frac{1}{2}); \alpha_l, \beta_l), \quad px(1) + qx'(1) = r_2(x(\frac{1}{2}); \alpha_l, \beta_l), \tag{3.20}$$

and for $k = 1, \dots, m$,

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = h_k(x(t_k), x'(t_k); \alpha_l, \beta_l, \alpha_l'), \tag{3.21}$$

and

$$x''(t) = J(t, x(t), x'(t); \beta_l, \beta_l'),\ t_k < t < t_{k+1},\ k = 0, \dots, m, \tag{3.22}$$

$$px(0) - qx'(0) = R_1(x(\frac{1}{2}); \beta_l), \quad px(1) + qx'(1) = R_2(x(\frac{1}{2}); \beta_l), \tag{3.23}$$

and for $k = 1, \dots, m$,

$$\Delta x(t_k) = u_k,$$

$$\Delta x'(t_k) = H_k(x(t_k), x'(t_k); \alpha_l, \beta_l, \beta_l'). \tag{3.24}$$

Assume that $\alpha_0 \leq \alpha_l \leq \beta_l \leq \beta_0$ for $l > 1$ and we will show that $\alpha_0 \leq \alpha_{l+1} \leq \beta_{l+1} \leq \beta_0$. Since α_l is a solution of (3.19)-(3.21) with $\beta_{l-1} \geq \alpha_l$, it follows from (3.1) that

$$\begin{aligned}
\alpha_l''(t) &= Q(t, \alpha_l(t), \alpha_l'(t); \alpha_{l-1}, \beta_{l-1}, \alpha_{l-1}') \\
&= f(t, \alpha_{l-1}(t), \alpha_{l-1}'(t)) + F_x(t, \beta_{l-1}(t))(\alpha_l(t) - \alpha_{l-1}(t)) \\
&\quad + \phi_1(t, \alpha_{l-1}(t), \alpha_{l-1}'(t)) - \phi_1(t, \alpha_l(t), \alpha_l'(t)), \\
&\geq f(t, \alpha_l(t), \alpha_l'(t)) + F_x(t, \alpha_l(t))(\alpha_{1-1}(t) - \alpha_l(t)) \\
&\quad + \phi_1(t, \alpha_l(t), \alpha_l'(t)) - \phi_1(t, \alpha_{l-1}(t), \alpha_{l-1}'(t)), \\
&\quad + F_x(t, \beta_{l-1}(t))(\alpha_l(t) - \alpha_{l-1}(t)) + \phi_1(t, \alpha_{l-1}(t), \alpha_{l-1}'(t)) \\
&\quad - \phi_1(t, \alpha_l(t), \alpha_l'(t)), \\
&\geq f(t, \alpha_l(t), \alpha_l'(t)) + (F_x(t, \alpha_l(t)) - F_x(t, \alpha_l(t)))(\alpha_{1-1}(t) - \alpha_l(t)) \\
&= f(t, \alpha_l(t), \alpha_l'(t)) = Q(t, \alpha_l(t), \alpha_l'(t); \alpha_l, \beta_l, \alpha_l').
\end{aligned}$$

Using (3.2), we obtain

$$\begin{aligned}
&p\alpha_l(0) - q\alpha_l'(0)\\
&= r_1(\alpha_l(\frac{1}{2}); \alpha_{l-1}, \beta_{l-1})\\
&= g_1(\alpha_{l-1}(\frac{1}{2})) + G_1'(\beta_{l-1}(\frac{1}{2}))(\alpha_l(\frac{1}{2}) - \alpha_{l-1}(\frac{1}{2}))\\
&\quad + \psi_1(\alpha_{l-1}(\frac{1}{2})) - \psi_1(\alpha_l(\frac{1}{2}))\\
&\le g_1(\alpha_l(\frac{1}{2})) + G_1'(\alpha_l(\frac{1}{2}))(\alpha_{l-1}(\frac{1}{2}) - \alpha_l(\frac{1}{2})) + \psi_1(\alpha_l(\frac{1}{2})) - \psi_1(\alpha_{l-1}(\frac{1}{2}))\\
&\quad + G_1'(\beta_{l-1}(\frac{1}{2}))(\alpha_l(\frac{1}{2}) - \alpha_{l-1}(\frac{1}{2})) + \psi_1(\alpha_{l-1}(\frac{1}{2})) - \psi_1(\alpha_l(\frac{1}{2}))\\
&\le g_1(\alpha_l(\frac{1}{2})) + (G_1'(\alpha_l(\frac{1}{2})) - G_1'(\alpha_l(\frac{1}{2})))(\alpha_{l-1}(\frac{1}{2}) - \alpha_l(\frac{1}{2}))\\
&= g_1(\alpha_l(\frac{1}{2})) = r_1(\alpha_l(\frac{1}{2}); \alpha_l, \beta_l).
\end{aligned}$$

Similarly, it can be shown that

$$p\alpha_l(1) + q\alpha_l'(1) \le r_2(\alpha_l(\frac{1}{2}); \alpha_l, \beta_l).$$

Using (3.3), the nondecreasing property of V_k' and the fact that $\beta_{l-1} \ge \alpha_l$, we obtain

$$\begin{aligned}
\Delta\alpha_l'(t_k) &= h_k(\alpha_l(t_k), \alpha_l'(t_k); \alpha_{l-1}, \beta_{l-1}, \alpha_{l-1}')\\
&= v_k(\alpha_{l-1}(t_k), \alpha_{l-1}'(t_k)) + V_k'(\beta_{l-1}(t_k))(\alpha_l(t_k) - \alpha_{l-1}(t_k))\\
&\quad - \phi_k(\alpha_l(t_k), \alpha_l'(t_k)) + \phi_k(\alpha_{l-1}(t_k), \alpha_{l-1}'(t_k))\\
&\ge v_k(\alpha_l(t_k), \alpha_l'(t_k)) + V_k'(\alpha_l(t_k))(\alpha_{l-1}(t_k) - \alpha_l(t_k))\\
&\quad - \phi_k(\alpha_{l-1}(t_k), \alpha_{l-1}'(t_k)) + \phi_k(\alpha_l(t_k), \alpha_l'(t_k))\\
&\quad + V_k'(\beta_{l-1}(t_k))(\alpha_l(t_k) - \alpha_{l-1}(t_k)) - \phi_k(\alpha_l(t_k), \alpha_l'(t_k))\\
&\quad + \phi_k(\alpha_{l-1}(t_k), \alpha_{l-1}'(t_k))\\
&= v_k(\alpha_l(t_k), \alpha_l'(t_k)) + (V_k'(\beta_{l-1}(t_k)) - V_k'(\alpha_l(t_k)))(\alpha_l(t_k) - \alpha_{l-1}(t_k))\\
&\ge v_k(\alpha_l(t_k), \alpha_l'(t_k)) = h_k(\alpha_l(t_k), \alpha_l'(t_k); \alpha_l, \beta_l, \alpha_l'),\ k = 1, \ldots, m.
\end{aligned}$$

Thus, α_l is a lower solution of (3.19)-(3.21). Similarly, we can show that β_l is an upper solution of (3.19)-(3.21). Hence, by Theorem 2.2, there exists a solution α_{l+1} of (3.19)-(3.21) satisfying

$$\alpha_0 \le \cdots \le \alpha_l \le \alpha_{l+1} \le \beta_l \le \cdots \le \beta_0.$$

In the same manner, we can show that

$$\alpha_0 \le \cdots \le \alpha_l \le \beta_{l+1} \le \beta_l \le \cdots \le \beta_0,$$

where β_{l+1} is a solution of (3.22)-(3.24) and α_l, β_l are lower and upper solutions of (3.22)-(3.24). Making use of the earlier arguments, it can be shown that α_{l+1}

and β_{l+1} are respectively lower and upper solutions of the BVP with impulse (1.1)-(1.3) and by Theorem 2.1, it follows that $\alpha_{l+1} \le \beta_{l+1}$. Hence, we arrive at the conclusion

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_l \le \alpha_{l+1} \le \beta_{l+1} \le \beta_l \le \cdots \le \beta_1 \le \beta_0.$$

We now apply Kamke convergence criterion to show that each of the two sequences $\{\alpha_l\}$ and $\{\beta_l\}$ converges in B to x, the unique solution of the BVP with impulse (1.1)-(1.3).

The Kamke Convergence Theorem does not apply directly to either the sequence $\{\alpha_l\}$ or $\{\beta_l\}$ as neither Q_l nor J_l converges uniformly on compact sets to f. In order to check this, we note that

$$Q_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(x_1 - \alpha_l(t)) + F(t, \alpha_l(t)) - F(t, x_1),$$

and

$$J_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(x_1 - \beta_l(t)) + F(t, \beta_l(t)) - F(t, x_1).$$

Define

$$\widehat{Q}_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(\alpha_{l+1}(t) - \alpha_l(t)) + F(t, \alpha_l(t)) - F(t, \alpha_{l+1}(t)),$$

$$\widehat{J}_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(\beta_{l+1}(t) - \beta_l(t)) + F(t, \beta_l(t)) - F(t, \beta_{l+1}(t)),$$

so that α_{l+1} is the unique solution of the BVP with impulse (1.1)-(1.3) with $f = \widehat{Q}_l$ and each $v_k = h_{kl},\ g_i = r_{il},\ i = 1, 2$. The Kamke convergence Theorem now does apply. Employing the arguments used in the proof of Theorem 2.2, we conclude that the sequence $\{\alpha_l\}$ converges in B to x, the unique solution of (1.1)-(1.3). Similarly, the sequence $\{\beta_l\}$ converges in B to the unique solution x of the BVP with impulse (1.1)-(1.3).

Now, we show the quadratic convergence of the sequences. For that, let us define $\gamma_n(t) = \beta_n(t) - x(t)$, $\chi_n(t) = x(t) - \alpha_n(t)$ and set $e_n = \max\{\|\gamma_n\|_B, \|\chi_n\|_B\}$. We will only prove the quadratic convergence of the sequence $\{\gamma_n\}$ as that of $\{\chi_n\}$is similar. Using (A_2), we find that

$$\begin{aligned} p\gamma_{n+1}(0) - q\gamma'_{n+1}(0) &\le g_1(\beta_{n+1}(\frac{1}{2})) - g_1(x(\frac{1}{2})) \\ &= g'_1(\eta_1(\frac{1}{2}))\gamma_{n+1}(\frac{1}{2}) \le \gamma_{n+1}(\frac{1}{2}), \end{aligned}$$

$(x(\frac{1}{2}) \le \eta_1(\frac{1}{2}) \le \beta_{n+1}(\frac{1}{2}))$ which can alternatively be written as

$$\gamma'_{n+1}(0) \ge \frac{p}{q}\gamma_{n+1}(0) - \frac{1}{q}\gamma_{n+1}(\frac{1}{2}) \ge 0 \text{ if } p \ge \frac{\gamma_{n+1}(\frac{1}{2})}{\gamma_{n+1}(0)}.$$

Also, for $x(\frac{1}{2}) \le \eta_2(\frac{1}{2}) \le \beta_{n+1}(\frac{1}{2})$, we have

$$\begin{aligned} p\gamma_{n+1}(1) + q\gamma'_{n+1}(1) &\le g_2(\beta_{n+1}(\frac{1}{2})) - g_2(x(\frac{1}{2})) \\ &= g'_2(\eta_2(\frac{1}{2}))\gamma_{n+1}(\frac{1}{2}) \le \gamma_{n+1}(\frac{1}{2}), \end{aligned}$$

which implies that

$$\gamma'_{n+1}(1) \le \frac{1}{q}\gamma_{n+1}(\frac{1}{2}) - \frac{p}{q}\gamma_{n+1}(1) \le 0 \text{ if } p \ge \frac{\gamma_{n+1}(\frac{1}{2})}{\gamma_{n+1}(1)}.$$

Hence we conclude that $\gamma'_{n+1}(0) \ge 0$ and $\gamma'_{n+1}(1) \le 0$ when $p = \max\{\frac{\gamma_{n+1}(\frac{1}{2})}{\gamma_{n+1}(0)}, \frac{\gamma_{n+1}(\frac{1}{2})}{\gamma_{n+1}(1)}\}$. For $t \in \bigcup_{k=0}^{m}(t_k, t_{k+1})$, the repeated application of the mean value theorem yields

$$\begin{aligned}
\gamma''_{n+1}(t) &= \beta''_{n+1}(t) - x''(t)\\
&= J(t, \beta_{n+1}(t), \beta'_{n+1}(t); \beta_n, \beta'_n) - f(t, x(t), x'(t))\\
&= f(t, \beta_n(t), \beta'_n(t)) + F_x(t, \beta_n(t))(\beta_{n+1}(t) - \beta_n(t))\\
&\quad - \phi_1(\beta_{n+1}(t), \beta'_{n+1}(t)) + \phi_1(t, \beta_n(t), \beta'_n(t)) - F(t, x(t))\\
&\quad + \phi_1(t, x(t), x'(t))\\
&= F(t, \beta_n(t)) - \phi_1(t, \beta_n(t), \beta'_n(t)) + F_x(t, \beta_n(t))(\beta_{n+1}(t) - \beta_n(t))\\
&\quad - \phi_1(t, \beta_{n+1}(t), \beta'_{n+1}(t)) + \phi_1(t, \beta_n(t), \beta'_n(t)) - F(t, x(t))\\
&\quad + \phi_1(t, x(t), x'(t))\\
&= F(t, \beta_n(t)) - F(t, x(t)) + F_x(t, \beta_n(t))(\beta_{n+1}(t) - \beta_n(t))\\
&\quad + \phi_1(t, x(t), x'(t)) - \phi_1(t, \beta_{n+1}(t), \beta'_{n+1}(t))\\
&= F_x(t, \xi_1(t))\gamma_n(t) - F_x(t, \beta_n(t))\gamma_n(t) + F_x(t, \beta_n(t))\gamma_{n+1}(t)\\
&\quad - \phi_{1x}(t, \xi_2(t), \xi_3(t))\gamma_{n+1}(t) - \phi_{1x'}(t, \xi_2(t), \xi_3(t))\gamma'_{n+1}(t),
\end{aligned}$$

where $x(t) \le \xi_1(t) \le \beta_n(t)$, $x(t) \le \xi_2(t) \le \beta_{n+1}(t)$ and $x'(t) \le \xi_3 \le \beta'_{n+1}(t)$. Thus, there exists $\xi_1(t) \le \xi_4(t) \le \beta_n(t)$ such that

$$\begin{aligned}
\gamma''_{n+1}(t) &= F_{xx}(t, \xi_4(t))\gamma_n(t)(\xi_1(t) - \beta_n(t))\\
&\quad + (F_x(t, \beta_n(t)) - \phi_{1x}(t, \xi_2(t), \xi_3(t)))\gamma_{n+1}(t)\\
&\quad - \phi_{1x'}(t, \xi_2(t), \xi_3(t))\gamma'_{n+1}(t)\\
&\ge -F_{xx}(t, \xi_4(t))\gamma_n^2(t) + f_{x'}(t, \xi_2(t), \xi_3(t))\gamma'_{n+1}(t),
\end{aligned}$$

where we have used the monotonicity of F_x. In particular, there exist $M > 0$ such that

$$\gamma''_{n+1}(t) - f'_x(t, \xi_2(t), \xi_3(t))\gamma'_{n+1}(t) \ge -Me_n^2, \tag{3.25}$$

where $M > \max_i \max_{(t,x)\in D_i} F_{xx}(t, x)$, and for $i = 0, \dots, m$,

$$D_i = \{(t, x) : t_i \le t \le t_{i+1}, \alpha_0(t) \le x \le \beta_0(t)\}.$$

Similarly, there exist appropriate ξ_5 and ξ_6 such that

$$\Delta\gamma'_{n+1}(t_k) - v_{kx'}(\xi_5(t_k), \xi_6(t_k))\gamma'_{n+1}(t_k) \ge -Me_n^2, \; k = 1, \dots, m. \tag{3.26}$$

Multiplying both sides of (3.25) by $\mu(t)$, we get

(3.27)

$$\mu(t)\gamma''_{n+1}(t) - \mu(t)f_{x'}(t,\xi_2(t),\xi_3(t))\gamma'_{n+1}(t) = (\gamma'_{n+1}(t)\mu(t))' \geq -M\mu(t)e_n^2,$$

where $\mu(t)$ denote the integrating factor associated with (3.25) and is given by

$$\mu(t) = \exp\left(-\int_0^t f_{x'}(s,\xi_2(s),\xi_3(s))ds\right).$$

Integrating both sides of (3.27) from t to 1, we get

$$\gamma'_{n+1}(1)\mu(1) - \gamma'_{n+1}(t)\mu(t) \geq -Me_n^2\int_t^1 \mu(s)ds,\ t_m \leq t \leq 1.$$

In view of $\gamma'_{n+1}(1) \leq 0$, the above expression reduces to

$$\gamma'_{n+1}(t) \leq Me_n^2\int_t^1 \frac{\mu(s)}{\mu(t)}ds.$$

Since γ_{n+1} converges to 0 in B, it ultimately follows that $(s,\zeta_2(s),\zeta_3(s))$ belongs to

$$\widehat{D} = \{(s,x_1,x_2) : t_m \leq s \leq 1, \alpha_0(s) \leq x_1 \leq \beta_0(s), x'(s)-1 \leq x_2 \leq x'(s)+1\}.$$

Thus, we can bound $\mu(t)$ away from both 0 and ∞ for sufficiently large n. In particular, for $t_m \leq t \leq 1$, there exists $N_1 > 0$ such that

$$\gamma'_{n+1}(t) \leq N_1e_n^2. \tag{3.28}$$

Applying (3.26) at t_m, we obtain

$$\begin{aligned}&\Delta\gamma'_{n+1}(t_m) - v_{mx'}(\xi_5(t_m),\xi_6(t_m))\gamma'_{n+1}(t_m)\\ = \ &\gamma'_{n+1}(t_m^+) - \gamma'_{n+1}(t_m) - v_{mx'}(\zeta_6(t_m),\zeta_7(t_m))\gamma'_{n+1}(t_m) \geq -Me_n^2.\end{aligned}$$

In view of (A_3) and taking $M_1 > 0$ as the bound for $v_{mx'}$ away from both 0 and ∞, we get

$$\gamma'_{n+1}(t_m^-) \geq -M_1e_n^2. \tag{3.29}$$

Using (3.27) and (3.29), there exists some $N_2 > 0$ such that $\gamma'_{n+1}(t) \leq N_2e_n^2$ for $t_{m-1} \leq t \leq t_m$. Arguing as before, we can find a suitable form of (3.29) at t_{m-1}. Continuing this procedure inductively, we can find $N > 0$ for sufficiently large n such that

$$\gamma'_{n+1}(t) \leq Ne_n^2,\ t \in \bigcup_{k=0}^{m}[t_k,t_{k+1}]. \tag{3.30}$$

Since $\gamma_{n+1}(t) \geq 0$ and $\gamma_{n+1} \in C[0,1]$, we integrate (3.30) from 0 to t and for sufficiently large n, we have $\gamma_{n+1}(t) - \gamma_{n+1}(0) \leq Ne_n^2$. As $\gamma_{n+1}(0) \geq 0$, therefore

$$0 \leq \gamma_{n+1}(t) \leq Ne_n^2. \tag{3.31}$$

Integrating (3.27) from 0 to t ($\leq t_1$), we obtain

$$\gamma'_{n+1}(t)\mu(t) - \gamma'_{n+1}(0) \geq -Me_n^2 \int_0^t \mu(s)ds.$$

As $\gamma'_{n+1}(0) \geq 0$, it follows that for $0 \leq t \leq t_1$, there exists $N_1 > 0$ such that

$$\gamma'_{n+1}(t) \quad \geq \quad -Me_n^2 \int_0^t \frac{\mu(s)}{\mu(t)} ds \geq -N_1 e_n^2$$

for sufficiently large n which is analogous to (3.28). In a similar manner, we can choose N large enough such that for $t \in \bigcup_{k=0}^{m} [t_k, t_{k+1}]$ and for sufficiently large n, we have

$$\gamma'_{n+1}(t) \geq -N_1 e_n^2. \tag{3.32}$$

From the estimates (3.30), (3.31) and (3.32), it follows that the sequence $\{\beta_n\}$ converges to x quadratically in B. This completes the proof. □

4. Concluding remarks

We have developed the generalized quasilinearization method for a class of impulsive three-point boundary value problems with nonlinear boundary conditions. It has been shown that the concavity assumption on the nonlinearities in the boundary conditions can be relaxed on the pattern similar to the one employed to relax the convexity condition on nonlinearity involved in the differential equation. Several interesting results appear as a special case of the work established in this paper, for example, if we take $p = 1$, $q = 0$, $g_1(x(\frac{1}{2})) = a$ and $g_2(x(\frac{1}{2})) = b$, the results of [17] appear as a special case of our problem. In case we take $g_1(x(\frac{1}{2})) = a$ and $g_2(x(\frac{1}{2})) = b$, our results correspond to an impulsive two-point boundary value problem with separated boundary conditions. Moreover, the classical method of quasilinearization for impulsive nonlinear three-point boundary value problems can be recorded by taking $\phi(t, x) \equiv 0$ and $\psi_i(x) \equiv 0$ in (A_1) and (A_2) of Theorem 3.1.

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